Origin of normal stress differences in rapid granular flows

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A method for performing a Chapman-Enskog-like expansion of the Boltzmann equation corresponding to granular gases is presented. A calculation of the stress tensor corresponding to a two-dimensional gas of inelastically colliding smooth disks serves to demonstrate the method. This calculation provides an answer to the long sought source of the normal stress differences in granular fluids. It turns out that, like in molecular fluids, this effect is second (Burnett) order in the shear rate but, unlike in simple molecular fluids, it is a sizeable effect; as such it can be considered as a measurable manifestation of the Burnett correction for simple fluids. [S1063-651X(96)04210-9]

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In recent years there has been a significant increase in the interest in the properties of granular systems [1–6]. These systems, which are of immense industrial importance, exhibit a variety of unusual properties. When strongly forced (e.g., sheared), granular systems can be completely fluidized; this state is coined ''rapid granular flow.'' One of the prominent properties of granular gases is the sizeable normal stress differences these systems exhibit when in a sheared state [1,7,8]. The question of the source of this effect has preoccupied a number of researchers [9,10]. Some investigators who employed kinetic theoretical methods for the study of granular flows merely stated that their theories could not account for this phenomenon, e.g., [11–13]. Jenkins and Richman [9] obtained anisotropic normal stresses by conjecturing a form of the single particle distribution function. Another theory for normal stress differences, presented in [10], proposes that density gradients are responsible for this phenomenon. In the latter work, this effect is attributed to (Enskog) corrections to the Boltzmann equation.

The similarity of the microscopic dynamics of rapid granular flows to that of molecular fluids has prompted numerous studies of granular systems which are based on the kinetic theory of gases [9,11–15]. The relevance of a kinetic approach to (at least dilute) rapid granular flows [which is supported by molecular dynamics (MD) results and successes of kinetic theories] can be appreciated by considering the quasielastic limit. In this limit the energy loss (due to inelasticity) in each collision can be small enough so that the time scale for local equilibration (typically, a few collisions per particle) is shorter than the time scale for energy decay (by inelastic collisions); consequently, one expects a near Maxwellian (local) distribution to develop (in quasielastic systems). In most previous investigations (see, however, [13]) the single particle distribution corresponding to granular systems has not been systematically derived from the corresponding Boltzmann equation—instead, various moment closures have been invoked [9,11,12,14,15]; clearly a systematic perturbative solution of the pertinent Boltzmann equation is called for. There is however a problem in constructing a Chapman-Enskog (CE) expansion for a granular system: due to the inelastic nature of the collisions the only steady state of such systems, in the absence of external forcing, is one of zero granular temperature—and the latter cannot serve as a ''zeroth order'' in a perturbation theory at finite granular temperatures. In this paper it is shown how this problem can be resolved. It is known [16–18], that the temperature T, of a homogeneous sheared granular system is proportional to γ^2/ε, where γ is the shear rate and ε is a measure of the degree of inelasticity (defined as $1-e^2$ where e is the coefficient of normal restitution). Consider the double limit $\gamma\to 0$ and $\varepsilon\to 0$ while γ^2/ε (or T) is fixed. In this limit one obtains an equilibrium system (at any predetermined temperature). On the basis of this observation it can be shown [19] that a perturbative expansion for the solution of the Boltzmann equation corresponding to a steady sheared state can be constructed by employing $\sqrt{\varepsilon}$ as a small parameter and considering the shear rate γ to be $O(\sqrt{\varepsilon})$. This expansion is limited to steady states alone. Below we present a generalization of this approach, which is achieved by considering γ and ε to be separate (small) expansion parameters. Consider, e.g., the Boltzmann equation for a (dilute) gas of hard disks in a plane, whose collisions are characterized by a single constant coefficient of normal restitution [13,20]

$$\frac{\partial f}{\partial t}+\mathbf{v}_1\cdot\boldsymbol{\nabla} f=\frac{\sigma_T}{2}\int_{\hat{\mathbf{k}}\cdot\mathbf{v}_{12}>0} d\hat{\mathbf{k}}\, d\mathbf{v}_2(\hat{\mathbf{k}}\cdot\mathbf{v}_{12})\times\left(\frac{1}{e^2}f(\mathbf{v}_1^*)f(\mathbf{v}_2^*)-f(\mathbf{v}_1)f(\mathbf{v}_2)\right)\equiv\mathcal{B}(f,f). \tag{1}$$

In Eq. (1), f denotes the single particle distribution function (in the above integral only the velocity dependence is spelled out), σ_T is the total cross section (equal to twice the diameter of a disk), e is the coefficient of normal restitution, $\hat{\mathbf{k}}$ is a unit vector pointing from the center of disk 1 to that of disk 2 at contact, $\mathbf{v}_1^*,\mathbf{v}_2^*$ and $\mathbf{v}_1,\mathbf{v}_2$ are the velocities of the colliding particles before and after the collision, respectively. The nonlinear Boltzmann collision operator $\mathcal{B}(f,f)$ is defined in Eq. (1). In the derivation below we specialize, for simplicity, to the case of homogeneous number density n, and homogeneous granular temperature [21]. Denote the inverse ''granular temperature'' by β where $\beta^{-1}(t)\equiv\langle u^2\rangle$. The brackets $\langle\,\rangle$ denote averaging with respect to f. Next, define a dimensionless single particle distribution function f, by $f \equiv n\beta\tilde{f}(\sqrt{\beta}\mathbf{u})$, where $\mathbf{u}$ is the fluctuating velocity (actual velocity minus the average velocity at a given point). Notice that $\tilde{f}$ is a space independent function of $\mathbf{u}$, in the homogeneous case. In the case of a simple shear flow field, $\mathbf{V}=\gamma y\hat{\mathbf{x}}$, Eq. (1) can be written in the following nondimensional form:

$$\tilde{\dot{\beta}}\left(\tilde{f}+\frac{1}{2}\tilde{\mathbf{u}}\cdot\frac{\partial\tilde{f}}{\partial\tilde{\mathbf{u}}}\right)-\tilde{\gamma}\tilde{u}_y\frac{\partial\tilde{f}}{\partial\tilde{u}_x}=\tilde{\mathcal{B}}(\tilde{f},\tilde{f}), \tag{2}$$

where $\tilde{\dot{\beta}}\equiv\dot{\beta}l/\sqrt{\beta}$, $\tilde{\gamma}\equiv\gamma l\sqrt{\beta}$, $\tilde{\mathbf{u}}\equiv\sqrt{\beta}\mathbf{u}$, $l=1/n\sigma_T$ is the mean free path, and

$$\tilde{\mathcal{B}}(\tilde{f},\tilde{f})\equiv\frac{1}{2}\int_{\hat{\mathbf{k}}\cdot\mathbf{u}_{12}>0}d\hat{\mathbf{k}}\,d\tilde{\mathbf{u}}_2(\hat{\mathbf{k}}\cdot\tilde{\mathbf{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\mathbf{u}}_1^*)\tilde{f}(\tilde{\mathbf{u}}_2^*)-\tilde{f}(\tilde{\mathbf{u}}_1)\tilde{f}(\tilde{\mathbf{u}}_2)\right). \tag{3}$$

Below we omit the tilde signs, with the understanding that all quantities are dimensionless unless otherwise specified. Next, we specialize to the case of near-elastic collisions: $\varepsilon\ll 1$, and small shear: $\gamma\ll 1$. Consider the expansion of f and $\dot{\beta}$ in powers of these parameters

$$f(\mathbf{u})=f_0(u)(1+\varepsilon\Phi_{01}+\gamma\Phi_{10}+\varepsilon^2\Phi_{02}+\gamma\varepsilon\Phi_{11}+\gamma^2\Phi_{20}+\cdots), \tag{4}$$

where $f_0(u)\equiv\exp(-u^2)/\pi$, and (since $\dot{\beta}$ vanishes in the absence of *both* inelasticity and shear)

$$\dot{\beta}=\varepsilon\dot{\beta}_{01}+\gamma\dot{\beta}_{10}+\varepsilon^2\dot{\beta}_{02}+\gamma\varepsilon\dot{\beta}_{11}+\gamma^2\dot{\beta}_{20}+\cdots . \tag{5}$$

This two parameter expansion is a generalization of the CE expansion for the case of rapid granular flows. The substitution of Eq. (4) and Eq. (5) in the Boltzmann equation yields a perturbative expansion for f in powers of ε and γ. At $O(\gamma)$ one obtains

$$\mathcal{L}\Phi_{10}=\dot{\beta}_{10}(1-u^2)+2u_xu_y, \tag{6}$$

where $\mathcal{L}$ is the standard linearized Boltzmann operator

$$\mathcal{L}\Phi\equiv\frac{1}{2\pi}\int_{\hat{\mathbf{k}}\cdot\mathbf{u}_{12}>0}d\hat{\mathbf{k}}\,d\mathbf{u}_2(\hat{\mathbf{k}}\cdot\mathbf{u}_{12})e^{-u_2^2}[\Phi(\mathbf{u}_1^*)+\Phi(\mathbf{u}_2^*)-\Phi(\mathbf{u}_1)-\Phi(\mathbf{u}_2)]. \tag{7}$$

Equation (7) is defined with $e=1$. The solubility condition for Eq. (6) requires the right-hand side (rhs) of this equation to be orthogonal to the eigenfunctions of $\mathcal{L}$, which have vanishing eigenvalues, i.e., to 1, $\mathbf{u}$, and u^2 (with a Maxwellian weight function [22]). This implies $\dot{\beta}_{10}=0$. The solution of Eq. (6) is therefore (due to the isotropy of $\mathcal{L}$) of the form $\Phi_{10}=2\hat{\Phi}_{10}(u)u_xu_y=\hat{\Phi}_{10}(u)u^2\sin 2\theta$, where (u,θ) are the polar coordinates of the vector $\mathbf{u}$ and $\hat{\Phi}_{10}$ is a function of the speed, which can be determined numerically [19] or (less accurately) by using, e.g., an expansion in Sonine polynomials [22]. At $O(\varepsilon)$ one obtains

$$\mathcal{L}\Phi_{01}=\dot{\beta}_{01}(1-u^2)+Q_1(u). \tag{8}$$

The function $Q_1(u)$ is the expansion of $\mathcal{B}(f_0,f_0)/f_0$ to first order in ε, given by [19]:

$$Q_1(u)=2\sqrt{\pi}e^{-u^2/2}\left[\frac{1}{8}\left(1-\frac{2}{3}u^4\right)I_0\left(\frac{u^2}{2}\right)+\frac{1}{12}u^2(1-u^2)I_1\left(\frac{u^2}{2}\right)\right], \tag{9}$$

where I_0 and I_1 are the zeroth and first order modified Bessel functions, respectively. The rhs of Eq. (8) is orthogonal to 1 and $\mathbf{u}$. The requirement that it is also orthogonal to u^2 yields $\dot{\beta}_{01}=\sqrt{\pi/8}$. The coefficient $\dot{\beta}_{01}$ determines the rate of cooling, due to the inelasticity of the collisions, to lowest order in ε. Notice that, unlike Eq. (6), Eq. (8) is specific to the CE expansion for granular fluids. Reverting to dimensional quantities, it follows that:

$$\frac{d}{dt}\langle u^2\rangle_{\text{cooling}}=-\sqrt{\pi/8}\varepsilon n\sigma_T\langle u^2\rangle^{3/2}+\text{higher order terms}. \tag{10}$$

Equation (10) is in conformity with the phenomenological result $\dot{T}_{\text{cooling}}\propto-\varepsilon n\sigma_T T^{3/2}$, for homogeneous systems [16–18]. Since the rhs of Eq. (8) depends on u alone, the solution is a function $\hat{\Phi}_{01}(u)$ of the speed which can be determined numerically to the desired accuracy [19]. The equation at $O(\gamma^2)$ is

$$\mathcal{L}\Phi_{20}=\dot{\beta}_{20}(1-u^2)+2u_xu_y\Phi_{10}-u_y\frac{\partial\Phi_{10}}{\partial u_x}-\Psi(\Phi_{10}), \tag{11}$$

where

$$\Psi(\Phi)\equiv\frac{1}{2\pi}\int_{\hat{\mathbf{k}}\cdot\mathbf{u}_{12}>0}d\hat{\mathbf{k}}\,d\mathbf{u}_2(\hat{\mathbf{k}}\cdot\mathbf{u}_{12})e^{-u_2^2}[\Phi(\mathbf{u}_1^*)\Phi(\mathbf{u}_2^*)-\Phi(\mathbf{u}_1)\Phi(\mathbf{u}_2)]. \tag{12}$$

Note that in Eq. (12), $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$ are defined with $e=1$. It is evident, by considering the form of Φ_{10} and symmetry considerations, that $\Psi(\Phi_{10})$ satisfies the solubility conditions. The other terms on the rhs of Eq. (11) satisfy the first and second solubility conditions. The third solubility condition determines $\dot{\beta}_{20}$ [19]: $\dot{\beta}_{20}=\int_0^\infty x^5e^{-x^2}\hat{\Phi}_{10}(x)dx\approx-0.8146$. The coefficient β_{20} determines, to lowest nontrivial order in γ, the heating caused by the shear. Reverting to dimensional quantities one obtains

$$\frac{d}{dt}\langle u^2\rangle_{\text{heating}}\approx 0.8146 l\gamma^2\sqrt{\langle u^2\rangle}+\text{higher order terms}. \tag{13}$$

It can be shown [19] that the rhs of Eq. (11) is of the form $A(u)+B(u)u^2\cos 2\theta+C(u)u^4\cos 4\theta$, hence the solution of Eq. (11) assumes the form: $\Phi_{20}=\hat{\Phi}_{20}^{(0)}(u)+\hat{\Phi}_{20}^{(2)}(u)u^2\cos 2\theta+\hat{\Phi}_{20}^{(4)}(u)u^4\cos 4\theta$, where $\hat{\Phi}_{20}^{(0)}$, $\hat{\Phi}_{20}^{(2)}$, and $\hat{\Phi}_{20}^{(4)}$ are scalar functions of u that can be determined numerically [19]. To lowest nontrivial order in the above perturbative expansion, the steady-state condition reads: $\dot{\beta}_{01}\varepsilon+\dot{\beta}_{20}\gamma^2=0$ (our results hold, of course, for transient dynamics—as well). The resulting (dimensional) relation between shear, inelasticity, and granular temperature, under steady-state conditions, is $\gamma \approx 0.8771\sqrt{\varepsilon\langle u^2\rangle}/l$. This result is in conformity with the mean field (qualitative) relation $T \propto \gamma^2/\varepsilon$. The contributions to f of $O(\gamma\varepsilon)$ and $O(\varepsilon^2)$ are responsible for an inelastic correction to the viscosity and to a next order correction to the inelastic cooling rate, respectively. These corrections are not considered in the present paper. Having found the form of the function f to first order in ε and second order in γ, we can now evaluate the stress tensor $\tau_{ij} = \langle u_i u_j \rangle$ to the same order. Since f_0 is (standardly) defined in such a way that the normalization, mean velocity, and temperature are given by its appropriate moments, it follows that the isotropic parts of the corrections to f_0 do not contribute to the diagonal components of the stress tensor. Hence, the isotropic parts of Φ_{ij} (e.g., $\hat{\Phi}_{01}$ and $\hat{\Phi}_{20}^{(0)}$) do not contribute to these components. Clearly (since $u_i u_j$ contains up to second harmonics in θ and, as mentioned, the operator $\mathcal{L}$ is isotropic), the fourth order harmonic does not contribute to the stress tensor as well. Consequently, only second order harmonics, i.e., terms proportional to $\cos 2\theta$ and $\sin 2\theta$, contribute to the stress tensor. The components of the stress tensor are obtained by a direct integration of the terms multiplying the above mentioned harmonics, the result being

$$\tau_{xx} = \tfrac{1}{2} + a\gamma^2, \quad \tau_{yy} = \tfrac{1}{2} - a\gamma^2, \quad \text{and} \quad \tau_{xy} = \tau_{yx} = b\gamma,$$

where [19]

$$a = \frac{1}{2}\int_0^\infty x^5 e^{-x^2} \hat{\Phi}_{20}^{(2)}(x)\,dx \approx 0.3395 \quad \text{and}$$

$$b = \frac{1}{2}\int_0^\infty x^5 e^{-x^2} \hat{\Phi}_{10}(x)\,dx \approx -0.4073.$$

Since $\tau_{xx} > \tau_{yy}$ we obtain a normal stress difference. Notice that this effect is (qualitatively) a consequence of the shear and not of the inelasticity. Reverting to dimensional quantities it follows that:

$$\frac{\tau_{xx}}{\tau_{yy}} \approx \frac{1 + 0.679\gamma^2 l^2/\langle u^2\rangle}{1 - 0.679\gamma^2 l^2/\langle u^2\rangle}. \tag{14}$$

Equation (14) is formally identical to the result one could have obtained by substituting the two-dimensional (2D) Burnett correction (the 3D Burnett formulas are given, e.g., in [22]) for the normal stresses. This formal resemblance is somewhat misleading since: (i) the CE expansion of the Boltzmann equation corresponding to a granular gas is *a priori* undefined, as explained in the above. (ii) The $\varepsilon \to 0$ limit is not a trivial limit, as one can realize by considering the fact that the only steady state of a sheared elastically colliding system (of infinite extent) is one of infinite temperature due to the continual heating by the shear, whereas a granular system under similar conditions has a genuine steady state. (iii) Equation (14) is only a lowest order (in γ and ε) expression of a more general result (which is ε dependent) which follows from the (above) generalization of the CE idea to inelastically colliding systems. All in all, naive usage of the Burnett results, while yielding the correct answer to lowest order, is not justified; a careful analysis of the corresponding Boltzmann equation is required. The ratio of the normal stresses increases with the mean free path when all the other parameters are fixed. Notice that in *steady granular systems the temperature is not predetermined but it is fixed by ε and γ*. Substitution of the steady-state relation between T and γ in Eq. (14) yields a normal stress ratio which is a universal function of ε alone. The result, to the presently calculated order in perturbation theory, is

$$\frac{\tau_{xx}}{\tau_{yy}} \approx \frac{1 + 0.522\varepsilon}{1 - 0.522\varepsilon}. \tag{15}$$

The latter function tends to unity as $\varepsilon \to 0$ and thus one may erroneously conclude that the normal stress difference is a feature of inelasticity. However, when $\varepsilon \to 0$ and γ is kept fixed, it follows from Eq. (14) that the normal stress difference remains intact [the lowest order at which the inelasticity influences the normal stress ratio is $O(\gamma^2\varepsilon)$]. The reason for the possible confusion is the fact that in the *steady* (sheared) *state* of a granular system $\gamma^2 \propto \varepsilon$ for a fixed value of the (granular) temperature (a result of the balance between collisional inelastic cooling and viscous heating), hence, one cannot separate the $\varepsilon \to 0$ and $\gamma \to 0$ limits in this case.

The unobservability of the normal stress difference in simple molecular fluids can be appreciated by noting that $\gamma^2 l^2/\langle u^2\rangle = O(10^{-21})$ for air at 20 °C, atmospheric pressure and $\gamma = 1\ \text{sec}^{-1}$. Only under extreme conditions (very cold and strongly sheared dilute gases) one stands the chance of observing a slight normal stress difference in simple molecular fluids. In granular fluids, however, this quantity is $O(\varepsilon)$ and amenable to measurement. *Thus the specific nature of granular fluids*, i.e., *the fact that $T \propto \gamma^2/\varepsilon$ renders the Burnett correction significant and observable in these systems*; *one may regard this effect in granular fluids as a measurable manifestation of the Burnett correction*. At this point we wish to mention again the Jenkins-Richman ansatz [9] of an anisotropic Maxwellian distribution for a steady sheared granular flow: in their theory (which compares favorably with simulations) a steady-state shear flow was considered and the resulting normal stress difference is $O(\varepsilon)$. Their results for the numerical coefficients ''a'' and ''b'' (in our notation) are close to ours [19]. The normal stress ratio predicted by Eq. (15) for $e = 0.8$, is $\tau_{xx}/\tau_{yy} \approx 1.463$. The numerical result of Walton and Braun [7] is 1.484 and that calculated by Jenkins and Richman is 1.439 (in the dilute limit). The difference between the results of Jenkins and Richman and our own is due to the fact that they use Enskog's equation to obtain a closure while we have performed a systematic CE-like expansion (their theory corresponds to effectively replacing the functions Φ_{ij} by constants). At this point it is worthwhile mentioning that normal stress differences are known in polymeric systems, nontrivial molecular systems, and in strongly sheared systems in general. As mentioned in the above, the existence of a very weak normal stress difference in sheared simple molecular fluids, can be deduced from the well known values of the Burnett coefficients for such systems. However, as we have hopefully shown in the above, the existence of *strong* normal stress differences in simple granular systems is a subtle issue: unlike polymeric systems these systems are isotropic on the molecular level and the CE expansion for the corresponding Boltzmann equation cannot straightforwardly be read off the corresponding CE expansion for elastic systems.

In summary, we have shown how one can perform a Chapman-Enskog analysis of the Boltzmann equation corresponding to granular systems. We carried it out to Burnett order and discovered that this order is the source of the (strong) normal stress differences observed in granular flows (with corrections due to higher orders). Since a similar (but weak) effect exists in simple molecular fluids one may state that the Burnett correction is a universal source of normal stress differences and that granular fluids provide a measurable manifestation of this effect.

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